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September 24, 2012

The goal of this note is to describe the action of pseudo-differential operators on the space $\mathcal{H}$ of L^2 functions which are harmonic outside a smooth closed hyper-surface Z of a compact Riemannian manifold without boundary (X, g) and whose traces from both sides of Z coïncide. We will represent these L^2 harmonic functions as harmonic extensions of functions in the Sobolev space $H^{-1/2}(Z)$ by a Poisson operator $\mathcal{P}$. The main result says that, if A is a pseudo-differential operator of degree $d < 3$, the operator

$$B = \mathcal{P}^\star \circ A \circ \mathcal{P}$$

is a pseudo-differential operator on Z of degree $d-1$ whose principal symbol of degree $d-1$ can be computed by integration of the principal symbol of A on the co-normal bundle of Z.

These "bilateral" extensions are simpler (at least for the Laplace operator) than the "unilateral" ones whose study is the theory of pseudo-differential operators on manifolds with boundary (see [1, 2, 3, 4, 6]).

1 Symbols

The following classes of symbols are defined in the books [4], sec. 7.1, and in [5], sec. 18.1. A *symbol of degree* d on $U_x \times \mathbb{R}^n_\xi$ where U is an open set in $\mathbb{R}^N$ is a smooth complex valued function $a(x, \xi)$ on $U \times \mathbb{R}^n$ which satisfies the following estimates: for any multi-indices (α, β), there exists a constant $C_{\alpha,\beta}$ so that

$$|D_x^\alpha D_\xi^\beta a(x;\xi)| \leq C_{\alpha,\beta}(1 + \|\xi\|)^{d-|\beta|} .$$

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The symbol a is called *classical* if a admits an expansion $a \sim \sum_{l=0}^{\infty} a_{d-l}$ where a_j is homogeneous of degree j (j an integer) for $\xi \in \mathbb{R}^n$ large enough; more precisely, for any $J \in \mathbb{N}$, $a - \sum_{j=0}^{J} a_{d-j}$ is a symbol of degree $d - J - 1$.

We will need the

Lemma 1 *If $a(x;\xi,\eta)$ is a symbol of degree $d < -1$ defined on $U_x \times \left(\mathbb{R}^n_\xi \times \mathbb{R}_\eta\right)$, $b(x;\xi) = \int_{\mathbb{R}} a(x;\xi,\eta)d\eta$ is a symbol of degree $d+1$ defined on $U_x \times \mathbb{R}^n_\xi$. Moreover, if a is classical, b is also classical and the homogeneous components of b are given for $l \leq d+1$, by $b_l(x;\xi) = \int_{\mathbb{R}} a_{l-1}(x;\xi,\eta)d\eta$*

2 A general reduction Theorem for pseudo-differential operators

We choose local coordinates in some neighborhood of a point in Z denoted $x = (z, y) \in \mathbb{R}^{d-1} \times \mathbb{R}$, so that $Z = \{y = 0\}$. We denote by $(\Omega_j,\ j = 1, \cdots N)$ a finite cover of Z by such charts and denote by Ω_0 an open set disjoint from Z so that $X = \cup_{j=0}^{N} \Omega_j$. We choose the charts Ω_j so that the densities $|dz|$ and $|dx|$ are the Lebesgue measures.

If X is a smooth manifold, we denote by $\mathcal{D}'(X)$ the space of generalized functions on X of which the space of smooth functions on X is a dense subspace. We assume that X and Z are equipped with smooth densities $|dx|$ and $|dz|$. This allows to identify generalized functions with Schwartz distributions, i.e. linear functionals on test functions; this duality extending the L^2 product is denoted by $\langle | \rangle$. We introduce the extension operator $\mathcal{E} : \mathcal{D}'(Z) \to \mathcal{D}'(X)$ sending the distribution f to the distribution $f\delta(y = 0)$ defined

$$\langle f\delta(y = 0)|\phi(z, y)\rangle = \langle f|\phi(z, 0)\rangle$$

and its adjoint, the trace $\mathcal{T} : C^\infty(X) \to C^\infty(Z)$ defined by $\phi \to \phi_{|Z}$. Let A be a pseudo-differential operator on X: let us call A_j the restriction of A to test functions compactly supported in Ω_j. We will work with one of the A_j's given by the following "quantization" rule

$$A_j u(z, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(\langle z-z'|\zeta\rangle + (y-y')\eta)} a_j(z, y; \zeta, \eta) u(z', y') dz'dy'd\zeta d\eta\ .$$

So we have formally, using the facts that the densities on X and Z are given by the Lebesgue measures in these local coordinates:

$$\mathcal{T} \circ A_j \circ \mathcal{E} v(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d-1}} e^{i\langle z-z'|\zeta\rangle} a_j(z, 0; \zeta, \eta) v(z') dz'd\zeta d\eta\ ,$$

which we can rewrite

$$\mathcal{T} \circ A_j \circ \mathcal{E} v(z) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d}} e^{i\langle z-z'|\zeta\rangle} b_j(z; \zeta) v(z') dz'd\zeta\ ,$$

with

$$b_j(z;\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}} a_j(z,0;\zeta,\eta)d\eta \ . \tag{1}$$

We have the

Theorem 1 *If A is a pseudo-differential operator on X of degree $m < -1$ whose full symbol in the chart Ω_j is a_j, then the operator $\mathcal{T}\circ A\circ\mathcal{E}$ is a pseudo-differential operator on Z of degree $m+1$ whose symbol is given in the charts $\Omega_j \cap Z$ by Equation (1).*

This is proved by looking at the actions on test functions compactly supported in the chart Ω_j, $j \geq 1$: then we use Lemma 1.

Remark 1 *The principal symbol can be described in a more intrinsic way: let $z \in Z$ be given, from the smooth densities on T_zX and on T_zZ given by $|dx|$ and $|dz|$, we get, using the Liouville densities, densities on the dual bundles $T_z^\star Z$ and $T_z^\star X$. Let us denote by $\Omega^1(E)$ the 1-dimensional space of densities on the vector space E. From the exact sequence*

$$0 \to N_z^\star Z \to T_z^\star X \to T_z^\star Z \to 0 \ ,$$

we deduce

$$\Omega^1(T^\star X) \equiv \Omega^1(N^\star Z)\otimes\Omega^1(T^\star Z)$$

and a canonical density $dm(z)$ in $\Omega^1(N_z^\star Z)$. The principal symbol of $B = \mathcal{T}\circ A\circ\mathcal{E}$ is given in coordinates by $b(z,\zeta) = (1/2\pi)\int_{N_z^\star Z} a(z;\zeta,\eta)dm(\eta)$.

3 The "bilateral" Dirichlet-to-Neumann operator

We will assume that the local coordinates $x = (z,y)$ along Z are chosen so that $g(z,0) = h(dz) + dy^2$ and the Riemannian volume along Z is $|dx|_g = |dz|_h|dy|$. We will choose the associated densities on X and Z. We will denote by Δ_g the Laplace-Beltrami operator on (X,g) as defined by Riemannian geometers (i.e. with a minus sign in front of the second order derivatives).

If f is given on Z, let us denote by $\mathcal{DN}(f)$ minus the sum of the interior normal derivatives on both sides of Z of the harmonic extension F of f; this always makes sense, even if the normal bundle of Z is not orientable. We have the

Lemma 2 *The distributional Laplacian of the harmonic extension F of a smooth function f on Z is $\Delta_g F = \mathcal{E}(\mathcal{DN}(f))$.*

Proof.–

The proof is a simple application of the Green's formula: by definition of the action of the Laplacian on distributions, if ϕ is a test function on X, $\langle \Delta_g F | \phi \rangle := \langle F | \Delta_g \phi \rangle$. We can compute the righthandside integral as an integral on $X \setminus Z$ using Green's formula.

$$\int_{X \setminus Z} (F \Delta_g \phi - \phi \Delta_g F) |dx|_g = \int_Z (F \delta \phi - \phi \delta F) |dz|_h$$

where δ is the sum of the interior normal derivatives from both sides of Z. Using the fact that $\Delta_g F = 0$ in $X \setminus Z$ and $\delta \phi = 0$, we get the result.

□

Denoting by Δ_g^{-1} the "quasi-inverse" of Δ_g defined by $\Delta_g^{-1} \phi_j = \lambda_j^{-1} \phi_j$ for the eigenfunctions ϕ_j of Δ_g with non-zero eigenvalue λ_j and $\Delta_g^{-1} 1 = 0$, we have $f = \left(\mathcal{T} \circ \Delta_g^{-1} \circ \mathcal{E} \right) \circ \mathcal{DN}(f)$ (mod constants). By Theorem 1, the operator $B = \mathcal{T} \circ \Delta_g^{-1} \circ \mathcal{E}$ is an elliptic self-adjoint pseudo-differential operator on Z. The operator $\mathcal{DN}$ is a right inverse of B modulo smoothing operators and hence also a left inverse modulo smoothing operators. So that $\mathcal{DN} = B^{-1}$ is an elliptic self-adjoint of principal symbol the inverse

$$\frac{1}{2\pi} \int_{\mathbb{R}} (\|\zeta\|_h^2 + \eta^2)^{-1} d\eta = \frac{1}{2\|\zeta\|_h} ,$$

namely $2\|\zeta\|_h$. Hence

Theorem 2 *The bilateral Dirichlet-to-Neumann$\mathcal{DN}$ is a self-adjoint elliptic pseudo-differential operator of degree* 1 *on* $L^2(Z, |dz|)$ *and of principal symbol* $2\|\zeta\|_h$. *The kernel of* $\mathcal{DN}$ *is the space of constant functions.*

The full symbol of $\mathcal{DN}$ can be computed in a similar way from the full symbol of the resolvent Δ_g^{-1} along Z.

4 The Poisson operator

Let A be an pseudo-differential operator on X of principal symbol a. We are interested to the restriction to the space $\mathcal{H}$ of the quadratic form $Q_A(F) = \langle AF | F \rangle$ associated to A. We will parametrize $\mathcal{H}$ as harmonic extensions of functions which are in $H^{-\frac{1}{2}}(Z)$ by the so-called Poisson operator denoted by $\mathcal{P}$; the pull-back R_A of Q_A on $L^2(Z)$ is defined by

$$R_A(f) = \langle A\mathcal{P}f | \mathcal{P}f \rangle = \langle \mathcal{P}^\star A \mathcal{P} f | f \rangle \ .$$

The goal of this section is to compute the operator $B = \mathcal{P}^\star A \mathcal{P}$ associated to the quadratic form R_A.

From Lemma 2, we have, modulo smoothing operators,

$$\mathcal{P} = \Delta_g^{-1} \circ \mathcal{E} \circ \mathcal{DN} \ .$$

Hence

$$B = \mathcal{DN} \circ \left[\mathcal{T} \circ \left(\Delta_g^{-1} \circ A \circ \Delta_g^{-1}\right) \circ \mathcal{E}\right] \circ \mathcal{DN} \ .$$

The operator $\Delta_g^{-1} \circ A \circ \Delta_g^{-1}$ is a pseudo-differential operator of principal symbol $a/(\|\zeta\|_h^2 + \eta^2)^2$ near Z.

Applying Theorem 1 to the inner bracket and Theorem 2, we get the:

Theorem 3 *If A is a pseudo-differential operator of degree $d < 3$ on X and $\mathcal{P}$ the Poisson operator associated to Z, the operator $B = \mathcal{P}^\star A \mathcal{P}$ is a pseudo-differential operator of degree $d-1$ on Z of principal symbol*

$$b(z, \zeta) = \frac{2}{\pi} \|\zeta\|_h^2 \int_{\mathbb{R}} \frac{a(z, 0; \zeta, \eta)}{(\|\zeta\|_h^2 + \eta^2)^2} d\eta \ .$$

Remark 2 *Note that if A is a pseudo-differential operator without the transmission property, the operator $A \circ \mathcal{P}$ may be ill-behaved and have disagreeable singularities along Z; however $\mathcal{P}^* A \mathcal{P}$ is always a good pseudo-differential operator on Z.*

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